

Cartan-Weyl 3-algebras and the BLG Theory II: Strong-Semisimplicity and Generalized Cartan-Weyl 3-algebras

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ABSTRACT: One of the most important questions in the Bagger-Lambert-Gustavsson (BLG) theory of multiple M2-branes is the choice of the Lie 3-algebra. The Lie 3-algebra should be chosen such that the corresponding BLG model is unitary and admits fuzzy 3-sphere as a solution. In this paper we propose a new condition: the Lie 3-algebras of use must be connected to the semisimple Lie algebras describing the gauge symmetry of D-branes via a certain reduction condition. We show that this reduction condition leads to a natural generalization of the Cartan-Weyl 3-algebras introduced in [1]. Similar to a Cartan-Weyl 3-algebra, a generalized Cartan-Weyl 3-algebra processes a set of step generators characterized by non-degenerate roots. However, its Cartan subalgebra is non-abelian in general. We give reasons why having a non-abelian Cartan subalgebra may be just right to allow for fuzzy 3-sphere solution in the corresponding BLG models. We propose that generalized Cartan-Weyl 3-algebras is the right class of metric Lie 3-algebras to be used in the BLG theory.

KEYWORDS: D-Branes, M-Theory, Gauge symmetry, Lie n -algebra.

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1. Introduction

This paper is a continuation of our paper [1]. In [1], we have introduced the concept of a *Cartan-Weyl 3-algebra*. A Lie 3-algebra is Cartan-Weyl if it admits a Cartan-Weyl basis of generators which consists of Cartan subalgebra of mutually commuting generators and a set of step generators that are characterized by a root space of degenerate 2-forms. We have also analysed the consistency conditions arising from the fundamental identity and obtained a complete classification of Cartan-Weyl 3-algebras.

In the case of Lie algebras, the existence of a Cartan-Weyl basis is equivalent to the fact that the Lie algebra is semisimple, i.e. the Lie algebra has no nonzero solvable ideals. For Lie 3 (or higher n)-algebras, due to the possibilities of having a number of different

notions of solvability and semisimplicity (see section 3 below), it is not clear whether any of these notions of semisimplicity is a good characterization of a Cartan-Weyl 3-algebra. One of the motivation of this paper is to address this question.

From the physicist point of view, the use of semisimple Lie algebras is both natural and extremely important since semisimple Lie algebras are the Lie algebras of compact connected Lie groups, natural devices for describing continuous symmetries. The story is much less clear for Lie 3 (or higher n)-algebras since the corresponding concepts of an exponential map or a finite transformation have not been developed. As a result, it is not clear whether any of these existing notions of semisimplicity is relevant for the description of multiple M2-branes in the BLG models [2–5]. The clarification of this is another motivation of this work.

A key observation in our analysis will be that the theory of root space decomposition for Lie algebras can be readily generalized to Lie 3 (or higher n)-algebras. The root space decomposition provides the most convenient framework for analysing the conditions that characterize Cartan-Weyl 3-algebras: the Abelianess of the Cartan subalgebra and the non-degeneracy of the root space components. It turns out that none of the existing notions of semisimplicity is sufficient to characterize Cartan-Weyl 3-algebras, and a stronger notion of semisimplicity is needed.

In this paper, we will analysis this question and propose a new notion of semisimplicity. This new notion of semisimplicity is motivated by a very natural physical considerations. And as it turns out, it leads to a natural generalization of Cartan-Weyl 3-algebras, i.e. one which has a root space whose root components are non-degenerate and a Cartan subalgebra which is non-abelian in general. This generalization is useful because it not only includes the Cartan Weyl Lie 3-algebra as a sub-case, but it also allows us to bypass a no-go theorem obtained in [1], which states that a Cartan-Weyl 3-algebra, other than \mathcal{A}_4 itself, does not contain the simple Lie 3-algebra \mathcal{A}_4 as a sub-algebra. As a result, the corresponding BLG theories do not contain fuzzy S^3 in their (semiclassical) description [6, 7]. With the generalized Cartan-Weyl 3-algebras, this is however possible in general.

The organization of the paper is as follows. In section 2, we show that for a metric Lie algebra, the root decomposition is rather complicated in general. What one obtains as a consequence of the existence of an invariant metric is that the roots with nonzero norm are non-degenerate, see (2.18). However further simplifications occur with a semisimple Lie algebra and one arrives at a Cartan-Weyl basis. In section 3, we develop the theory of root decomposition for Lie n -algebras. We will also examine the various notions of semisimplicity existed in the literature and show that none of them help to simplify the structure of the root space decomposition. In section 4, we propose a new definition of semisimplicity that is motivated by consideration of a possible connection between the Lie 3-algebra symmetry of multiple M2-branes and the Lie algebra symmetry of multiple

D-branes system. We will show that this notion of semisimplicity leads to a natural generalization of the Cartan-Weyl 3-algebras. This is the main result of this paper. A generalized Cartan-Weyl 3-algebra is almost the same as a Cartan-Weyl 3-algebra, but with the essential difference that its Cartan subalgebra is non-abelian in general. The paper is ended with some further discussions in section 5.

2. Structural Theory of Lie Algebras: Root Decomposition

2.1 Root decomposition of Lie algebras

As a result of its Lie algebraic structure, a Lie algebra always carry a decomposition characterized by a root space. It turns out the same construction can be carried over immediately to Lie n -algebras. So let us first review the theory of root space decomposition for Lie algebras.

2.1.1 Representation of nilpotent Lie algebras

Let $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation of the Lie algebra \mathfrak{g} and let $\lambda \in \mathfrak{g}^*$ be any linear form. Define

$$V^\lambda(\mathfrak{g}) := \{v \in V \mid (\rho(x) - \lambda(x)E)^m v = 0, \text{ for some } m > 0 \text{ and for all } x \in \mathfrak{g}\}, \quad (2.1)$$

where E is the identity operator. If $V^\lambda(\mathfrak{g}) \neq 0$, then $V^\lambda(\mathfrak{g})$ is said to be a root subspace of the representation ρ , and λ is called a weight. A particularly interesting result is obtained when \mathfrak{g} is nilpotent,

Theorem 2.1. *Let $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation of a nilpotent complex Lie algebra \mathfrak{g} . Then*

$$V = \bigoplus_{i=1}^s V^{\lambda_i}(\mathfrak{g}), \quad (2.2)$$

where $\lambda_i \in \mathfrak{g}^*$ are different weights of the representation ρ .

2.1.2 Weights and roots with respect to a nilpotent subalgebra

Let \mathfrak{g} be a complex Lie algebra, \mathfrak{h} a nonzero nilpotent subalgebra of it, and ρ a complex linear representation. Applying the above result to the representation $\rho|_{\mathfrak{h}}$, we obtain the decomposition

$$V = \bigoplus_{i=1}^s V^{\lambda_i}, \quad (2.3)$$

where the different linear forms $\lambda_i : \mathfrak{h} \rightarrow \mathbb{C}$ are different weights of the representation $\rho|_{\mathfrak{h}}$, and V^{λ_i} are the corresponding root subspaces. The space of *weights of the representation ρ with respect to \mathfrak{h}* will be denoted by $\Phi_\rho(\mathfrak{h}) = \{\lambda_1, \dots, \lambda_s\}$.

This result can be applied to the adjoint representation $\rho = \text{ad}$ of \mathfrak{g} . The fact that \mathfrak{h} is nilpotent implies that $\mathfrak{h} \subset \mathfrak{g}^0$ and so $0 \in \Phi_{\text{ad}}(\mathfrak{h})$. Nonzero weights are called *roots of the Lie algebra with respect to \mathfrak{h}* . Denote the system of all roots by $\Delta_{\mathfrak{g}}(\mathfrak{h}) = \Phi_{\text{ad}}(\mathfrak{h})/\{0\}$. We obtain:

Theorem 2.2. *Given a nilpotent subalgebra \mathfrak{h} of \mathfrak{g} , one has the decomposition of \mathfrak{g} as a vector space*

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \left(\bigoplus_{\alpha \in \Delta_{\mathfrak{g}}(\mathfrak{h})} \mathfrak{g}^{\alpha} \right). \quad (2.4)$$

This is called the root decomposition of \mathfrak{g} with respect to \mathfrak{h} . Here, explicitly it is,

$$\mathfrak{g}^0 = \{y \in \mathfrak{g} \mid (\text{ad } x)^m y = 0, \text{ for some } m > 0 \text{ and for all } x \in \mathfrak{h}\}, \quad (2.5)$$

$$\mathfrak{g}^{\alpha} = \{y \in \mathfrak{g} \mid (\text{ad } x - \alpha(x)E)^m y = 0, \text{ for some } m > 0 \text{ and for all } x \in \mathfrak{h}\}. \quad (2.6)$$

The root spaces $\mathfrak{g}^0, \mathfrak{g}^{\alpha}$ are naturally endorsed with a Lie-algebraic structure. We have:

Proposition 2.3.

$$[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}] \begin{cases} \subset \mathfrak{g}^{\alpha+\beta}, & \alpha + \beta \in \Phi_{\text{ad}}(\mathfrak{h}), \\ = 0, & \text{otherwise.} \end{cases} \quad (2.7)$$

Thus the subspace \mathfrak{g}^0 is a subalgebra of \mathfrak{g} .

2.1.3 Cartan subalgebras

The decomposition (2.4) of Lie algebra is general and applies for any nilpotent subalgebra \mathfrak{h} of \mathfrak{g} . A particularly interesting kind of nilpotent subalgebras is the Cartan subalgebra. A subalgebra \mathfrak{h} is called a *Cartan subalgebra* if it is nilpotent and equal to its normalizer¹. One can easily show that a Cartan subalgebra is a maximal nilpotent subalgebra but the converse is not true.

Back to the root decomposition (2.4), one can show that a nilpotent subalgebra \mathfrak{h} is a Cartan subalgebra iff $\mathfrak{g}^0 = \mathfrak{h}$. Therefore we obtain:

Theorem 2.4. *Let \mathfrak{g} be a complex Lie algebra and \mathfrak{h} be a Cartan subalgebra. One has the root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_{\mathfrak{g}}(\mathfrak{h})} \mathfrak{g}^{\alpha} \right). \quad (2.8)$$

We note on passing the following interesting relation:

¹The normalizer of a subspace V is defined by $N_{\mathfrak{g}}(V) = \{x \in \mathfrak{g} \mid [x, V] \subset V\}$

Proposition 2.5. *For the root decomposition (2.4) of a Lie algebra \mathfrak{g} with respect to its Cartan subalgebra \mathfrak{h} , the Killing metric on \mathfrak{h} can be expressed in terms of the roots as follows:*

$$\kappa(h, k) = \sum_{\alpha} n_{\alpha} \alpha(h) \alpha(k), \quad \text{for } h, k \in \mathfrak{h}. \quad (2.9)$$

Here $n_{\alpha} := \dim \mathfrak{g}^{\alpha} \geq 1$.

Proof. This follows from the definition of \mathfrak{g}^{α} that, when restricted to \mathfrak{g}^{α} , $\text{ad } h$, $\text{ad } k$ are of the form

$$\text{ad } h = \begin{pmatrix} \alpha(h) & & * \\ & \ddots & \\ 0 & & \alpha(h) \end{pmatrix}, \quad \text{ad } k = \begin{pmatrix} \alpha(k) & & * \\ & \ddots & \\ 0 & & \alpha(k) \end{pmatrix}, \quad (2.10)$$

where the matrices are n_{α} dimensional. \square

The above discussion leaves open the question of the existence of Cartan subalgebra. There is in fact a simple way to construct a Cartan subalgebra if the ground field contains sufficiently many elements, e.g. for the complex field \mathbb{C} . We will refer the readers to any classic textbook, e.g. [8, 9], for the details. We will be contented to state here:

Theorem 2.6. *There always exists a Cartan subalgebra for Lie algebra over \mathbb{C} ². Moreover all Cartan subalgebras of a complex Lie algebra \mathfrak{g} are conjugate under automorphism.*

Although the root decomposition (2.8) of Lie algebra is quite neat, the root spaces \mathfrak{g}^0 and \mathfrak{g}^{α} 's are still rather complicated in general. It will be nice to be able to obtain further information about their structure in special circumstances. The simplest situation one can imagine is that \mathfrak{g}^{α} is of dimension one for $\alpha \neq 0$ and that all the H_I 's commute with each other. In this case, if we denote the generator of \mathfrak{g}^{α} by E^{α} and the generators of \mathfrak{h} by H_I , $I = 1, \dots, N$ for some N , then it follows immediately from (2.6) and (2.7) that

$$\begin{aligned} [H_I, H_J] &= 0, \\ [H_I, E^{\alpha}] &= \alpha_I E^{\alpha}, \\ [E^{\alpha}, E^{\beta}] &= \begin{cases} 0, & \text{if } \alpha + \beta \neq 0 \text{ not a root,} \\ c(\alpha, \beta) E^{\alpha+\beta}, & \text{if } \alpha + \beta \neq 0 \text{ is a root,} \\ \in \mathfrak{h}, & \text{if } \alpha + \beta = 0. \end{cases} \end{aligned} \quad (2.11)$$

for some function $c(\alpha, \beta)$. Such a basis of generators $\{H_I, E^{\alpha}\}$ is called a *Cartan-Weyl basis* and the Lie algebra is called a *Cartan-Weyl Lie algebra*. The question to ask is when does such a basis of generators exists. Of course the answer is well known: a

²This is true as long as the ground field \mathbb{F} is algebraically closed and is of sufficiently large characteristic.

Cartan-Weyl basis of generators exists iff the Lie algebra is semisimple [8]. In the next subsections we will outline the main ideas of the proof. This exercise will also serve to help us finding the appropriate notion of semisimplicity for Lie 3 (or higher n)-algebras in order for something like the Cartan-Weyl basis to exist.

2.2 Properties of root space for metric Lie algebras

By definition, a metric on a Lie algebra is a bilinear symmetric form which is invariant. A Killing metric can always be constructed on any Lie algebra. In general, it is possible to have more than one metric on a Lie algebra. On simple Lie algebras, the Killing metric becomes the unique (up to a nonvanishing scaling factor) metric and is non-degenerate³.

In the main text of this paper, we are interested in the root space structure of a metric Lie 3-algebra. Therefore it is instructive to consider here first the more general case of a metric Lie algebra before restricting to the case of a semisimple Lie algebra. In particular this means we do not assume the form (2.9) of the metric, which holds only for the Killing metric.

Let \mathfrak{g} be a Lie algebra, \mathfrak{h} be a Cartan subalgebra and $\langle \cdot, \cdot \rangle$ be a non-degenerate metric. Then the following lemmas 2.7 - 2.10 hold. The proofs of these statements can be found in [8]. We note that they can be modified straightforwardly to work for any Lie algebra with a non-degenerate metric, not just with the Killing metric.

Lemma 2.7. *If α, β are any two weights and $\alpha + \beta \neq 0$, then $\mathfrak{g}^\alpha \perp \mathfrak{g}^\beta$ relative to the metric.*

Lemma 2.8. *The metric is non-degenerate when restricted to \mathfrak{h} . If α is a root, then $-\alpha$ is also a root. Moreover \mathfrak{g}^α and $\mathfrak{g}^{-\alpha}$ are dual spaces relative to the metric.*

Corollary. *The metric is non-degenerate when restricted to \mathfrak{h} .*

As a result of the corollary, we have that for any $\rho \in \mathfrak{h}^*$, there exists a unique $h_\rho \in \mathfrak{h}$ such that

$$\rho(h) = \langle h_\rho, h \rangle, \quad \text{for all } h \in \mathfrak{h}. \quad (2.12)$$

The mapping $\rho \rightarrow h_\rho$ is 1-1 and subjective. Moreover, (2.12) induces a non-degenerate bilinear form on \mathfrak{h}^* :

$$\langle \rho, \sigma \rangle := \langle h_\rho, h_\sigma \rangle, \quad \text{for } \rho, \sigma \in \mathfrak{h}^*. \quad (2.13)$$

It follows immediately that:

Lemma 2.9. *Let $e_\alpha \in \mathfrak{g}^\alpha$ be such that $[h, e_\alpha] = \alpha(h)e_\alpha$ for all $h \in \mathfrak{h}^*$, then it is*

$$[e_\alpha, e_{-\alpha}] = \langle e_\alpha, e_{-\alpha} \rangle h_\alpha, \quad (2.14)$$

³A metric is non-degenerate if $\langle x, y \rangle = 0$ for all y implies that $x = 0$.

for all $e_{-\alpha} \in \mathfrak{g}^{-\alpha}$. Here h_α is defined as in (2.12). Moreover due to 2.8, one can choose $\langle e_\alpha, e_{-\alpha} \rangle = 1$.

Proof. By the invariance of the metric, it is $\langle [e_\alpha, e_{-\alpha}], h \rangle = \langle e_{-\alpha}, [e_\alpha, h] \rangle = \langle e_{-\alpha}, e_\alpha \rangle \alpha(h)$. On the hand, (2.12) implies that $\langle \langle e_\alpha, e_{-\alpha} \rangle h_\alpha, h \rangle = \langle e_{-\alpha}, e_\alpha \rangle \langle h_\alpha, h \rangle = \langle e_{-\alpha}, e_\alpha \rangle \alpha(h)$. The claim follows from the non-degeneracy of the metric $\langle \cdot, \cdot \rangle$ when restricted to \mathfrak{h} . \square

Lemma 2.10. *If α is a root with nonvanishing norm, then $n_\alpha = \dim \mathfrak{g}^\alpha = 1$. Moreover the only integral multiples $k\alpha$ of α which are roots are $\alpha, 0, -\alpha$.*

Proof. For instruction, we outline the proof in [8] here. First of all, since the metric on \mathfrak{h} is non-degenerate, it follows immediately from (2.13) that $\langle \alpha, \alpha \rangle \neq 0$. Next consider $e_\alpha, e_{-\alpha}$ and h_α as defined in lemma 2.9 and set

$$\mathcal{R} = \mathbb{C}e_\alpha \oplus \mathbb{C}h_\alpha \oplus \sum_{k=1}^{\infty} \mathfrak{g}^{-k\alpha}, \quad (2.15)$$

then \mathcal{R} is an invariant subspace of \mathfrak{g} relative to $\text{ad}h, h \in \mathfrak{h}$ since

$$[h, e_\alpha] = \alpha(h)e_\alpha, \quad [h, h_\alpha] = 0, \quad [h, \mathfrak{g}^{-k\alpha}] \subset \mathfrak{g}^{-k\alpha}. \quad (2.16)$$

The restriction of $\text{ad}h$ to $\mathfrak{g}^{-k\alpha}$ has the single characteristic root $-k\alpha(h)$. Hence we have $\text{tr}_{\mathcal{R}}(\text{ad}h) = \alpha(h)[1 - n_{-\alpha} - 2n_{-2\alpha} - \dots]$. In particular

$$\text{tr}_{\mathcal{R}}(\text{ad}h_\alpha) = \langle \alpha, \alpha \rangle [1 - n_{-\alpha} - 2n_{-2\alpha} - \dots]. \quad (2.17)$$

Since \mathcal{R} is a subalgebra containing e_α and $e_{-\alpha}$, it is invariant under $\text{ad}e_\alpha$ and $\text{ad}e_{-\alpha}$; and since $h_\alpha = [e_\alpha, e_{-\alpha}]$, therefore $[\text{ad}e_\alpha, \text{ad}e_{-\alpha}] = \text{ad}h_\alpha$ and hence $\text{tr}_{\mathcal{R}}(\text{ad}h_\alpha) = 0$. The equation (2.17) has the only solution $n_{-\alpha} = 1, n_{-2\alpha} = \dots = 0$. Replacing α by $-\alpha$, we obtain our claim. \square

As a result of this lemma, we obtain

Theorem 2.11. *Let \mathfrak{g} be a Lie algebra carrying a non-degenerate metric and \mathfrak{h} is a Cartan subalgebra, we have the root decomposition for \mathfrak{g}*

$$\mathfrak{g} = \underbrace{\mathfrak{h}}_{\text{Non-abelian}} \oplus \underbrace{\left(\bigoplus_{\langle \alpha, \alpha \rangle \neq 0} \mathfrak{g}^\alpha \right)}_{\text{dimension 1}} \oplus \underbrace{\left(\bigoplus_{\langle \alpha, \alpha \rangle = 0} \mathfrak{g}^\alpha \right)}_{\text{dimension arb.}}, \quad (2.18)$$

where each of the root space \mathfrak{g}^α in the last factor is of dimension one. In general nothing can be said about the dimensions of the root space \mathfrak{g}^α with zero norm roots, i.e. $\langle \alpha, \alpha \rangle = 0$.

2.3 Properties of root space for semisimple Lie algebras

For a semisimple algebra, the Killing metric is not just non-degenerate, but also satisfies the important property (2.9). Using this one can establish the followings:

Lemma 2.12. *For a semisimple Lie algebra \mathfrak{g} with Cartan subalgebra \mathfrak{h} , it is*

$$\alpha([\mathfrak{h}, \mathfrak{h}]) = 0, \quad \text{for all } \alpha \in \mathfrak{h}^*. \quad (2.19)$$

Proof. To see how (2.9) can be put into use, let us show the proof of this lemma. We have seen above that for any $\alpha \in \mathfrak{h}^*$, there exists a $h_\alpha \in \mathfrak{h}$ such that $\alpha(h) = \langle h_\alpha, h \rangle$ where $\langle \cdot, \cdot \rangle$ is the Killing metric. Therefore for $h = [h_1, h_2]$, it is $\alpha(h) = \text{tr}([\text{ad } h_1, \text{ad } h_2] \text{ad } h_\alpha) = 0$ since the matrices $\text{ad } h_1$, $\text{ad } h_2$, $\text{ad } h_\alpha$ are all upper triangular as \mathfrak{h} is nilpotent and it is $\text{tr}(ABC) = \text{tr}(ACB)$ for any upper matrices A, B, C . \square

Proposition 2.13. *For a semisimple Lie algebra \mathfrak{g} , the Cartan subalgebra \mathfrak{h} is Abelian.*

Proof. Using again (2.9) and (2.19), we have $\langle h', k \rangle = 0$ for all $h' \in [\mathfrak{h}, \mathfrak{h}]$, $k \in \mathfrak{h}$. Now since the Killing metric is non-degenerate when restricted to \mathfrak{h} , therefore it must be $h' = 0$. \square

Proposition 2.14. *For a semisimple Lie algebra, it is*

$$\langle \alpha, \alpha \rangle \neq 0 \quad (2.20)$$

for every nonzero roots α relative to the Killing metric $\langle \cdot, \cdot \rangle$ on \mathfrak{h}^ .*

Proof. We refer the readers to section 4.1 of [8] for the proof of this proposition. \square

As a result, for semisimple Lie algebras, the factor of root spaces with zero norm roots is absent. Moreover, the Cartan subalgebra is Abelian. This establishes the existence of a Cartan-Weyl basis (2.11) for semisimple Lie algebras. Denoting the generator of each root space component \mathfrak{g}^α by E^α and the generators of the Cartan subalgebra by H_I . Suitably normalizing the generators, the Killing metric reads

$$\langle E^\alpha, E^\beta \rangle = \delta_{\alpha+\beta}, \quad \langle E^\alpha, H^I \rangle = 0, \quad g_{IJ} := \langle H_I, H_J \rangle, \quad (2.21)$$

where g_{IJ} , the restriction of the Killing metric on the Cartan subalgebra, is non-degenerate for a semisimple Lie algebra. The Lie brackets written in the Cartan-Weyl basis takes the form:

$$\begin{aligned} [H_I, H_J] &= 0, \\ [H_I, E^\alpha] &= \alpha_I E^\alpha, \\ [E^\alpha, E^\beta] &= \begin{cases} 0, & \text{if } \alpha + \beta \neq 0 \text{ not a root,} \\ c(\alpha, \beta) E^{\alpha+\beta}, & \text{if } \alpha + \beta \neq 0 \text{ is a root,} \\ -\alpha \cdot H & \text{if } \alpha + \beta = 0, \end{cases} \end{aligned} \quad (2.22)$$

where, $\alpha \cdot H = \alpha_I g^{IJ} H_J$ and g^{IJ} is the inverse of g_{IJ} . Here we have used the invariance of the metric in deriving the relation for $[E^\alpha, E^{-\alpha}]$.

The coefficient $c(\alpha, \beta)$ is antisymmetric in its arguments and satisfies the following conditions:

$$c(\alpha, \beta)c(\gamma, \alpha + \beta) + c(\beta, \gamma)c(\alpha, \beta + \gamma) + c(\gamma, \alpha)c(\beta, \gamma + \alpha) = 0 \quad (2.23)$$

if each of $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ and $\alpha + \beta + \gamma$ is a root; and

$$c(\alpha, \beta)c(\alpha + \beta, -\alpha) - c(-\alpha, \beta)c(-\alpha + \beta, \alpha) = \alpha \cdot \beta \quad (2.24)$$

if α, β are roots. These conditions follow from the Jacobi identities $[[E^\alpha, E^\beta], E^\gamma] = \dots$ and $[[E^\alpha, E^\beta], E^{-\alpha}] = \dots$. The second condition can also be written as

$$c(\alpha, \beta)c(-\alpha, -\beta) - c(\alpha, -\beta)c(-\alpha, \beta) = \alpha \cdot \beta \quad (2.25)$$

since $c(\alpha, \beta) = c(\beta, \gamma) = c(\gamma, \alpha)$ for roots such that $\alpha + \beta + \gamma = 0$.

The equations (2.23) and (2.25) constraint the root system and the coefficients $c(\alpha, \beta)$ algebraically. Solving these conditions gives a complete classification of semisimple Lie algebras [8].

3. Theory of Lie n -Algebras and Root Space Decomposition

3.1 Basic notions of semisimplicity

Let \mathcal{A} be a Lie n -algebra [10, 11]. A subspace $I \subset \mathcal{A}$ is an *ideal* if $[I\mathcal{A} \cdots \mathcal{A}] \subset I$. It is an *subalgebra* if $[II \cdots I] \subset I$. A Lie n -algebra is said to be *simple* if there is no proper ideal and $\dim \mathcal{A} > 1$. Simple Lie n -algebras have been classified and have a very simple structure.

Theorem 3.1. *A simple Lie n -algebra is isomorphic to one of the Lie n -algebra $\mathcal{A}_{p,q}$. The Lie n -algebra $\mathcal{A}_{p,q}$ is a metric Lie n -algebra with signature (p, q) , $p + q = n + 1$. It has $n + 1$ generators e_i , $i = 1, \dots, n + 1$ and is defined by the metric*

$$\langle e_i, e_j \rangle = \varepsilon_i \delta_{ij} \quad (3.1)$$

and the n -bracket relations

$$[e_1, \dots, \hat{e}_i, \dots, e_{n+1}] = (-1)^i \varepsilon_i e_i. \quad (3.2)$$

The signs ε_i are given by $(+\cdots+)$ for $\mathcal{A}_{0,n+1} := \mathcal{A}_{n+1}$, $(-+\cdots+)$ for $\mathcal{A}_{1,n}$, $(--+\cdots+)$ for $\mathcal{A}_{2,n-1}$ etc.

To discuss semisimplicity, one needs the concept of nilpotency. However as discussed by Kasymov [11], there is a spectrum of notions of nilpotency and solvability for Lie n -algebras.

Definition 3.1. An ideal I of an Lie n -algebra is called k -nilpotent if $I_k^r = 0$ for some $r \geq 0$. An ideal I of an Lie n -algebra is called k -solvable if $I_k^{(r)} = 0$ for some $r \geq 0$. Here $I_k^r, I_k^{(r)}$ are defined inductively by:

$$I_k^0 := I, \quad I_k^{s+1} := [I_k^s, \underbrace{I, \dots, I}_{k-1}, \underbrace{\mathcal{A}, \dots, \mathcal{A}}_{n-k}], \quad (3.3)$$

$$I_k^{(0)} := I, \quad I_k^{(s+1)} := [\underbrace{I_k^{(s)}, I_k^{(s)}, \dots, I_k^{(s)}}_k, \underbrace{\mathcal{A}, \dots, \mathcal{A}}_{n-k}] \quad (3.4)$$

and $k = 2, \dots, n$.

The concept of k -solvability (or k -nilpotency) for different k are distinct. Special cases are $k = n$ which is the one considered originally by Filippov [10] and $k = 2$ which is considered originally by Kuzmin [11]. It is clear that a k -solvable (respectively k -nilpotent) ideal I is also $(k+1)$ -solvable (respectively $(k+1)$ -nilpotent).

Definition 3.2. The largest k -solvable ideal of a finite dimensional Lie n -algebra \mathcal{A} is called the k -radical $\mathfrak{R}_k(\mathcal{A})$. A Lie n -algebra \mathcal{A} is called k -semisimple if $\mathfrak{R}_k(\mathcal{A}) = 0$.

It is obvious that if a Lie n -algebra is k -semisimple, then it is also $(k-1)$ -semisimple. In particular, for $n = 3$, a Lie 3-algebra which is semisimple in the sense of Filippov is also semisimple in the sense of Kuzmin. Ling [12] has shown that a Lie n -algebra semisimple in the Filippov sense can be written as a direct sum of its simple ideals. Therefore Lie 3-algebras which are Filippov-semisimple are too restrictive to be useful in the BLG models.

What about Kuzmin-semisimple Lie 3-algebras? It is easy to establish the followings:

Proposition 3.2. Consider the Lorentzian algebra [13] $\mathcal{A} = \mathfrak{g} \oplus \mathbb{C}(u, v)$, where \mathfrak{g} is a Lie algebra. \mathcal{A} has a metric which is given by an extension of the Killing metric on \mathfrak{g} to \mathcal{A} with the nonvanishing component $\langle u, v \rangle = 1$, and has the 3-brackets

$$[u, g_1, g_2] = [g_1, g_2]_{\mathfrak{g}}, \quad [g_1, g_2, g_3] = -\langle [g_1, g_2], g_3 \rangle_{\mathfrak{g}} v. \quad (3.5)$$

Then \mathcal{A} is solvable in the sense of Filippov. It is Kuzmin-semisimple (Kuzmin-solvable) iff the Lie algebra \mathfrak{g} is semisimple (solvable).

In the same way, it is easy to show that:

Proposition 3.3. The Lie 3-algebra with a maximally isotropic center [16, 17] is Kuzmin-semisimple iff the Lie algebra factors present there are semisimple.

Now it is known that BLG theories based on metric Lie 3-algebras with maximally isotropic center describes gauge theories on D2-branes. However there may still be other type of Kuzmin-semisimple Lie 3-algebras which is more appropriate for the description of multiple M2-branes. We will investigate this question below and propose a more specialised type of Kuzmin-semisimple Lie 3-algebras that is suitable for this purpose.

3.2 Root space decomposition

One of the powerful results about the structure of Lie algebras is the root space decomposition of Lie algebras with respect to a Cartan subalgebra. It is natural to ask if a similar result holds for Lie n -algebra. Let us examine it.

Let \mathcal{A} be a Lie n -algebra. A vector space V is called a Lie n \mathcal{A} -module if one can define a Lie n -algebra structure on the direct sum $\mathcal{B} := V + \mathcal{A}$ such that

$$[V, \mathcal{A}, \dots, \mathcal{A}] \subset V. \quad (3.6)$$

In this case one can associate to any set of elements $(a) = (a_1, \dots, a_{n-1}) \in \mathcal{A}^{\wedge(n-1)}$ a linear transformation $\rho_{(a)} = \rho_{(a_1, \dots, a_{n-1})}$ acting on V in accordance with the rule:

$$\rho_{(a)} : v \longmapsto [v, a_1, \dots, a_{n-1}]. \quad (3.7)$$

The operators $\rho_{(a)}$ satisfy the relations

$$[\rho_{(a)}, \rho_{(b)}] = \sum_{i=1}^{n-1} \rho_{(a_1, \dots, a_i R_{(b)}, \dots, a_{n-1})}, \quad (3.8)$$

$$\rho_{([a_1, \dots, a_n], b_2, \dots, b_{n-1})} = \sum_{i=1}^{n-1} \rho_{(a_i, b_2, \dots, b_{n-1})} \rho_{(a_1, \dots, \hat{a}_i, \dots, a_n)}, \quad (3.9)$$

where $R_{(a)}$ is the right multiplication $cR_{(a)} = [c, a_1, \dots, a_{n-1}]$. In this case we say there is a representation ρ of the Lie n -algebra \mathcal{A} on the space V . Note that the operators $\rho_{(a)}$ form a Lie algebra. We will denote by $L_\rho(\mathcal{A})$ the Lie algebra generated by the operators $\rho_{(a)}$. Note also that the operators $R_{(a)}$'s satisfy the same relations (3.8), (3.9), so it form a representation of \mathcal{A} on \mathcal{A} . This is called the *regular representation*. Using the regular representation R , one can introduce a *Killing form* for any Lie n -algebra by

$$\kappa((a); (b)) := \text{tr}(R_{(a)} R_{(b)}) \quad (3.10)$$

where $(a) = (a_1, \dots, a_{n-1})$, $(b) = (b_1, \dots, b_{n-1})$. The Killing form is invariant. Moreover it is known that [11, 14]:

Lemma 3.4. *Let \mathcal{A} be a Lie n -algebra, then*

- (Cartan criterion of semisimplicity) \mathcal{A} is semisimple in the Kuzmin sense iff the Killing form is non-degenerate,
- (Cartan criterion of solvability) \mathcal{A} is solvable in the Kuzmin sense iff $\kappa(c_1, a_2, \dots, a_{n-1}; c_2, b_2, \dots, b_{n-2}) = 0$ for all $c_1, c_2 \in \mathcal{A}^2$, $a_i, b_j \in \mathcal{A}$.

Let us denote the Lie algebra generated by the operators $R_{(a)}$ as $L(\mathcal{A})$. Kasymov [11] has shown that for a nilpotent (in the sense of Filippov) subalgebra \mathcal{H} of the Lie n -algebra \mathcal{A} , the subalgebra $L(\mathcal{H})$ generated by operators $R_{(h)}$, $(h) = (h_1, \dots, h_{n-1})$, $h_i \in \mathcal{H}$ is a nilpotent subalgebra of the Lie algebra $L(\mathcal{A})$. Therefore the theorem 2.2 for Lie algebra implies the following root space decomposition for a Lie n -algebra:

Theorem 3.5. *Let \mathcal{H} be a nilpotent subalgebra (in the sense of Filippov) of a Lie n -algebra \mathcal{A} , and $\rho = R|_{\mathcal{H}}$ is the regular representation of \mathcal{H} in \mathcal{A} . Then we have the decomposition*

$$\mathcal{A} = \mathcal{A}^0 \oplus \left(\bigoplus_{\alpha \in \Delta_{\mathcal{A}}(\mathcal{H})} \mathcal{A}^\alpha \right), \quad (3.11)$$

where

$$\begin{aligned} \mathcal{A}^0 &= \{y \in \mathcal{A} \mid yR_{(H)}^m = 0 \text{ for some } m > 0 \text{ \& for all } (H) \in \mathcal{H}^{\wedge(n-1)}\}, \\ \mathcal{A}^\alpha &= \{y \in \mathcal{A} \mid y(R_{(H)} - \alpha((H))E)^m = 0 \text{ for some } m > 0 \text{ \& for all } (H) \in \mathcal{H}^{\wedge(n-1)}\}. \end{aligned} \quad (3.12)$$

Here $\Phi_\rho(\mathcal{H}) = \{\lambda_1, \dots, \lambda_s\}$ is collection of the different weights $\lambda_i \in (\mathcal{H}^{\wedge(n-1)})^*$ of the representation ρ , and $\Delta_{\mathcal{A}}(\mathcal{H}) = \Phi_\rho(\mathcal{H})/\{0\}$ is the collection of all nonzero weights (called roots).

Note that roots are skew-symmetric in their argument and can be viewed as a $(n-1)$ -forms, generalizing the concept of roots in ordinary Lie algebra.

In Lie algebra theory, a Cartan subalgebra is defined as a subalgebra which is nilpotent and equal to its normalizer. Cartan subalgebras play a central role in the structure theory of finite dimensional Lie algebra. For example, as reviewed in section 2, the properties of Cartan subalgebras allow us to classify finite dimensional semisimple Lie algebras completely.

With the different notions of nilpotency and solvability to choose from for a Lie n -algebras (for $n \geq 3$), there is also a possibility of different notions of Cartan subalgebras. It turns out that the really useful definition is the one referring to nilpotency in the Filippov sense. Defining the *normalizer* of a subspace V of \mathcal{A} to be the subspace $\mathfrak{N}(V) = \{a \in \mathcal{A} \mid [a, V, \dots, V] \subset V\}$. We have the definition:

Definition 3.3. A subalgebra \mathcal{H} of a Lie n -algebra \mathcal{A} is a *Cartan subalgebra* if it is nilpotent in the sense of Filippov and equal to its own normalizer.

Obviously the dimension of a Cartan subalgebra of a Lie n -algebra is at least $n - 1$.

With this definition, Kasymov was able to establish the existence of Cartan subalgebras for any finite dimensional Lie n -algebra \mathcal{A} over \mathbb{C} [11, 15]. Moreover all Cartan subalgebras are conjugate to each other [15]. For example, for the Lorentzian 3-algebra (3.5), one easily see that the subalgebra defined by $\mathcal{H} := \{u, v, h_i\}$ is a Cartan subalgebra. Here h_i are the generators of the Cartan subalgebra of the Lie algebra \mathfrak{g} .

Back to theorem 3.5, one can show that [11] $\mathcal{A}^0 = \mathcal{H}$ if \mathcal{H} is a Cartan subalgebra. As a result, theorem 3.5 is refined to:

Theorem 3.6. *Let \mathcal{A} be a Lie n -algebra and \mathcal{H} be a Cartan subalgebra of it, then we have the root space decomposition*

$$\mathcal{A} = \mathcal{H} \oplus \left(\bigoplus_{\alpha \in \Delta_{\mathcal{A}}(\mathcal{H})} \mathcal{A}^{\alpha} \right). \quad (3.13)$$

The root space components satisfy the n -bracket

$$[\mathcal{A}^{\alpha_1}, \dots, \mathcal{A}^{\alpha_n}] \begin{cases} \subset \mathcal{A}^{\alpha_1 + \dots + \alpha_n}, & \alpha_1 + \dots + \alpha_n \in \Phi_R(\mathcal{H}) \\ = 0 & \text{otherwise} \end{cases} \quad (3.14)$$

Moreover, in analogous to Lie algebras, the Killing form on $\mathcal{H}^{(n-1)}$ can be expressed in terms of the roots

$$\kappa((h); (h')) = \sum_{\alpha} n_{\alpha} \alpha((h)) \alpha((h')). \quad (3.15)$$

This completes our review of the theory of Lie n -algebra known in the literature.

3.3 Properties of root spaces for metric Lie n -algebras

Next let us investigate further the properties of this root space decomposition for a Lie n -algebra when there is additionally a non-degenerate metric. Following essentially the same analysis as for Lie algebras, it is not difficult to show that:

Lemma 3.7. *Let \mathcal{A} be a Lie n -algebra and $\langle \cdot, \cdot \rangle$ a non-degenerate metric on \mathcal{A} . Then, (i) For any two weights α, β and $\alpha + \beta \neq 0$, $\mathcal{A}^{\alpha} \perp \mathcal{A}^{\beta}$ relative to the metric. (ii) The metric is non-degenerate when restricted to a Cartan subalgebra \mathcal{H} . (iii) If α is a root, then $-\alpha$ is also a root and \mathcal{A}^{α} and $\mathcal{A}^{-\alpha}$ are dual spaces relative to the metric.*

One can be quite explicit with the metric. If one denotes the generators of the Cartan subalgebra \mathcal{H} by H_I and the generators of the root space components \mathcal{A}^{α} by E_i^{α} , $i = 1, \dots, n_{\alpha}$ where $n_{\alpha} = \dim \mathcal{A}^{\alpha}$, i.e.

$$\{T^A\} = \{H_I, E_i^{\alpha}\}, \quad \text{where } I = 1, \dots, N, \quad \text{and } i = 1, \dots, n_{\alpha}, \quad (3.16)$$

then one can always choose the basis of generators such that the metric takes the form

$$\langle H_I, H_J \rangle = g_{IJ}, \quad \langle H_I, E_i^\alpha \rangle = 0, \quad \langle E_i^\alpha, E_j^\beta \rangle = \delta_{ij} \delta^{\alpha+\beta}, \quad (3.17)$$

where g_{IJ} is non-degenerate. To obtain a real Lie n -algebra, one can impose the Hermitian structure

$$(H_I)^\dagger = H_I, \quad (E_i^\alpha)^\dagger = E_i^{-\alpha}. \quad (3.18)$$

In this case, the index m is the number of negative eigenvalues of the metric g_{IJ} .

With the non-degenerate metric, one can associate to each root ρ and fixed $h_1, \dots, h_{n-2} \in \mathcal{H}$, a unique element $h_{(\rho; h_1, \dots, h_{n-2})} \in \mathcal{H}$ such that

$$\rho(h_1, \dots, h_{n-2}, h) = \langle h_{(\rho; h_1, \dots, h_{n-2})}, h \rangle, \quad \text{for all } h \in \mathcal{H}. \quad (3.19)$$

It is easy to establish a statement similar to the lemma 2.9 for Lie algebras:

Lemma 3.8. *Let $e_\alpha \in \mathcal{A}^\alpha$ be such that $[e_\alpha, h'_1, \dots, h'_{n-1}] = \alpha((h'))e_\alpha$ for all $(h') = (h'_1, \dots, h'_{n-1})$, $h'_i \in \mathcal{H}$, and let $e_{-\alpha} \in \mathcal{A}^{-\alpha}$, then*

$$[e_\alpha, e_{-\alpha}, h_1, \dots, h_{n-2}] = \langle e_{-\alpha}, e_\alpha \rangle h_{(\alpha; h_1, \dots, h_{n-2})} (-1)^n \quad (3.20)$$

for $h_1, \dots, h_{n-2} \in \mathcal{H}$.

At this point, one may then try to establish something like lemma 2.10 about the dimension of the root space components \mathcal{A}^α . But one sees an essential obstacle immediately. For Lie algebras, we have the relation $[h, h_\alpha] = 0$ for all $h \in \mathfrak{h}$. This follows from the definition of h_α and the Jacobi identity. This relation has played an essential role in establishing the lemma 2.10. For Lie n -algebra, by using the definition (3.20) and the fundamental identity, one can see that the relation $[h'_1, \dots, h'_{n-1}, h_{(\alpha; h_1, \dots, h_{n-2})}] = 0$ for arbitrary $h'_i, h_j \in \mathcal{H}$ is true only if \mathcal{H} is Abelian. Let us therefore ask when is \mathcal{H} Abelian.

We recall that in the case of a Lie algebra with a metric being non-degenerate on the Cartan subalgebra, one can show immediately that the Cartan subalgebra is Abelian if the condition $\alpha([\mathfrak{h}, \mathfrak{h}]) = 0$ holds (see proposition 2.13). This is so, for example, if the Lie algebra is semisimple (Lemma 2.12). For higher Lie n -algebra, one can similarly show the following:

Lemma 3.9. *Let \mathcal{A} be a Lie n -algebra and \mathcal{H} be a Cartan subalgebra. If*

i. the Killing form is non-degenerate on the subspace generated by (a_1, \dots, a_{n-1}) , where $a_i \in \mathcal{H}$,

ii. and

$$\alpha([h_1, \dots, h_n], h'_2, \dots, h'_{n-1}) = 0, \quad \text{for all } h_i, h'_j \in \mathcal{H}, \quad (3.21)$$

then \mathcal{H} is Abelian.

Proof. Using (3.15), we have $\kappa([h_1, \dots, h_n], \tilde{h}_2, \dots, \tilde{h}_{n-1}; \tilde{h}_1, \dots, \tilde{h}_{n-1}) = 0$ for arbitrary $h_i, \tilde{h}_j, \tilde{h}_k \in \mathcal{H}$. The stated non-degeneracy of the Killing form thus implies that $[h_1, \dots, h_n] = 0$. \square

Let us examine these two conditions more closely and ask when do these two conditions hold. For the first condition, we recall that the Killing form is non-degenerate iff the Lie n -algebra is semisimple in the Kuzmin sense. However even in this case, the non-degeneracy of the Killing form only implies (using its invariance) that the Killing form is non-degenerate when restricted on the subspace generated by (a_1, \dots, a_{n-1}) , where $a_i \in \mathcal{A}^{\lambda_i}$ and $\lambda_1 + \dots + \lambda_{n-1} = 0$. This is weaker than the first condition except for the case of $n = 2$ where the non-degeneracy of the Killing metric of a Lie algebra implies that it is also non-degenerate when restricted on a Cartan subalgebra. As for the second condition, we recall that for Lie algebras, the condition $\alpha([\mathfrak{h}, \mathfrak{h}]) = 0$ is a consequence of the fact that for each $\alpha \in \mathfrak{h}^*$, there exists a unique $h_\alpha \in \mathfrak{h}^*$ such that

$$\alpha(h) = \langle h_\alpha, h \rangle \quad (3.22)$$

for all $h \in \mathfrak{h}$ ⁴. For Lie n -algebras, the natural generalization of (3.22) is to require that for each $\alpha \in (\mathcal{H}^{\wedge(n-1)})^*$, there exists a unique $(h_\alpha) \in \mathcal{H}^{\wedge(n-1)}$ such that

$$\alpha((h)) = \kappa((h); (h_\alpha)), \quad (3.23)$$

where κ is the Killing form. Assuming its validity, we have therefore

$$\alpha([h_1, \dots, h_n], h'_2, \dots, h'_{n-1}) = \text{tr}(R_{([h_1, \dots, h_n], h'_2, \dots, h'_{n-1})} R_{(h_\alpha)}). \quad (3.24)$$

However this is nonvanishing since even though

$$R_{([h_1, \dots, h_n], h'_2, \dots, h'_{n-1})} = \sum_{i=1}^n (-1)^{i-1} R_{(h_i, h'_1, \dots, h'_{n-1})} R_{(h_1, \dots, \hat{h}_i, \dots, h_i)}, \quad (3.25)$$

and the order of R in the trace does not matter (this is because all the R 's involved are nilpotent matrices and one has $\text{tr}(abc) = \text{tr}(acb)$ for nilpotent matrices a, b, c), there is no complete cancellation as in the case for Lie algebras. Thus we conclude the condition (3.21) does not follow from (3.23). Instead, the condition (3.21) appears to be an independent condition that is of interest to be studied for its own right.

As a result, we conclude that the two conditions listed in the lemma 3.9 do not follow from k -semisimplicity. In general, for a k -semisimple metric Lie n -algebra \mathcal{A} ($k < n$),

⁴As a result, $\alpha([h_1, h_2]) = \langle h_\alpha, [h_1, h_2] \rangle = \text{tr}(\text{ad}[h_1, h_2] \text{ad } h) = \text{tr}([\text{ad } h_1, \text{ad } h_2] \text{ad } h) = 0$ since $\text{ad } h_1$, $\text{ad } h_2$ and $\text{ad } h$ are nilpotent matrices.

1. the Cartan subalgebra is non-abelian, and
 2. the root space components \mathcal{A}^α are generally of dimension greater than 1.
- i.e. the root space decomposition for a k -semisimple Lie n -algebra takes the form

$$\mathcal{A} = \underbrace{\mathcal{H}}_{\text{Non-abelian}} \oplus \underbrace{\left(\bigoplus_{\alpha \in \Delta_{\mathcal{A}}(\mathcal{H})} \mathcal{A}^\alpha \right)}_{\text{dimension arb.}}. \quad (3.26)$$

This is much more complicated than that of a semisimple Lie algebra.

4. Strong-Semisimplicity and Generalized Cartan-Weyl n -Algebras

When one of the worldvolume spatial dimension is compactified, a system of multiple M2-branes reduces to a system of multiple D2-branes whose worldvolume theory is equipped with a $U(N)$ gauge symmetry. Therefore in order for a BLG theory to describe multiple M2-branes, there should be a connection between the 3-algebra describing the symmetry of the multiple M2-branes and the (semisimple) Lie algebra describing the symmetry of multiple D-branes systems.

4.1 A reduction condition: strong-semisimplicity

Let us start with a simple observation about general Lie n -algebras. Consider a Lie n -algebra \mathcal{A} . Let us fix $\bar{h}_1, \dots, \bar{h}_{n-2} \in \mathcal{A}$ and consider a 2-bracket defined by

$$[x, y]_{\langle \bar{h} \rangle} := [x, y, \bar{h}_1, \dots, \bar{h}_{n-2}]. \quad (4.1)$$

Here we have used the symbol $\langle \bar{h} \rangle$ to denote the collection of the $(n-2)$ elements $\bar{h}_1, \dots, \bar{h}_{n-2}$. It is easy to see that the 2-bracket $[\cdot, \cdot]_{\langle \bar{h} \rangle}$ obeys the Jacobi identity as a result of the fundamental identity for the n -bracket. This allows one to introduce a Lie algebra structure on any Lie n -algebra. Let us denote the corresponding Lie algebra as $\mathcal{A}_{\langle \bar{h} \rangle} := (\mathcal{A}, [\cdot, \cdot]_{\langle \bar{h} \rangle})$. More generally, one can fix a collection of p elements $\bar{h}_1, \dots, \bar{h}_p \in \mathcal{A}$, $1 \leq p \leq n-2$, and consider the $(n-p)$ -bracket

$$[x_1, \dots, x_{n-p}]_{(\bar{h}_1, \dots, \bar{h}_p)} := [x_1, \dots, x_{n-p}, \bar{h}_1, \dots, \bar{h}_p], \quad x_i \in \mathcal{A}. \quad (4.2)$$

The algebra $\mathcal{A}_{(\bar{h}_1, \dots, \bar{h}_p)} := (\mathcal{A}, [\cdot, \dots, \cdot]_{(\bar{h}_1, \dots, \bar{h}_p)})$ form a Lie $(n-p)$ -algebra.

Let us denote the k -radical of the Lie $(n-p)$ -algebra $\mathcal{A}_{(\bar{h}_1, \dots, \bar{h}_p)}$ by $\mathfrak{R}_{k, (\bar{h}_1, \dots, \bar{h}_p)}(\mathcal{A})$. Obviously the k -radical is defined only for $k \leq n-p$.

Definition 4.1. Let \mathcal{A} be a Lie n -algebra \mathcal{A} and $2 \leq k \leq n$ fixed. We say that \mathcal{A} is (k, p) -semisimple, $0 \leq p \leq n-k$, if there exists $\bar{h}_1, \dots, \bar{h}_p \in \mathcal{A}$ such that $\mathfrak{R}_{k, (\bar{h}_1, \dots, \bar{h}_p)}(\mathcal{A}) = 0$, i.e. if the associated Lie $(n-p)$ -algebra $\mathcal{A}_{(\bar{h}_1, \dots, \bar{h}_p)}$ is semisimple.

It is easy to establish that:

Lemma 4.1. *Let \mathcal{A} be a Lie n -algebra and $2 \leq k \leq n$ fixed. Take $n - k$ elements $\bar{h}_1, \dots, \bar{h}_{n-k} \in \mathcal{A}$. For each $0 \leq p \leq n - k$, the k -radicals of the associated Lie $(n - p)$ -algebras $\mathcal{A}_{(\bar{h}_1, \dots, \bar{h}_p)}$ satisfy*

$$\mathfrak{R}_k(\mathcal{A}) \subset \mathfrak{R}_{k,(\bar{h}_1)}(\mathcal{A}) \subset \dots \subset \mathfrak{R}_{k,(\bar{h}_1, \dots, \bar{h}_{n-k})}(\mathcal{A}). \quad (4.3)$$

Proof. Denote $I = \mathfrak{R}_{k,(\bar{h}_1, \dots, \bar{h}_p)}$ for some fixed p . The fact that I is a maximal k -solvable ideal of \mathcal{A} implies that $I_k^{(r)} = 0$ for some r where we recall the definition of $I_k^{(r)}$:

$$I_k^{(r)} := [\underbrace{I_k^{(r-1)}, \dots, I_k^{(r-1)}}_k, \underbrace{\mathcal{A}, \dots, \mathcal{A}}_{n-p-k}, \underbrace{\bar{h}_1, \dots, \bar{h}_p}_p]. \quad (4.4)$$

By restricting one of the \mathcal{A} to be \bar{h}_{p+1} , we obtain

$$[\underbrace{I_k^{(r-1)}, \dots, I_k^{(r-1)}}_k, \underbrace{\mathcal{A}, \dots, \mathcal{A}}_{n-p-k-1}, \underbrace{\bar{h}_{p+1}, \bar{h}_1, \dots, \bar{h}_p}_{p+1}] = 0. \quad (4.5)$$

This implies that I is a k -solvable ideal of the Lie $(n - p - 1)$ -algebra $\mathcal{A}_{(\bar{h}_1, \dots, \bar{h}_p, \bar{h}_{p+1})}$ and so $I \subset \mathfrak{R}_{k,(\bar{h}_1, \dots, \bar{h}_p, \bar{h}_{p+1})} = 0$. Hence the claim. \square

As a result of this lemma, it is clear that (k, p) -semisimplicity implies $(k, p - 1)$ -semisimplicity. Therefore for fixed k , $(k, n - k)$ -semisimplicity is the strongest and $(k, 0)$ -semisimplicity is the weakest. Note that $(k, 0)$ -semisimplicity is the k -semisimplicity originally introduced by Kasymov, with $(2, 0)$ -semisimplicity being Kuzmin sense of semisimplicity and $(n, 0)$ -semisimplicity being the Filippov sense of semisimplicity. In view of this, one can introduce a graded notion of k -semisimplicity (with k fixed) for a Lie n -algebra and we have the following web of (k, p) -semisimplicity for Lie n -algebras. For a fixed $2 \leq K \leq n$, we have

$$\begin{array}{ccccccc} (K, 0) & \Leftarrow & (K, 1) & \cdots \Leftarrow & (K, n - K) & & \\ \Downarrow & & \Downarrow & & \Downarrow & & \\ (K - 1, 0) & \Leftarrow & (K - 1, 1) & \cdots \Leftarrow & (K - 1, n - K) & \Leftarrow & (K - 1, n - K + 1) \\ \vdots & & \vdots & & \vdots & & \vdots \quad \ddots \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ (2, 0) & \Leftarrow & (2, 1) & \cdots \Leftarrow & (2, n - K) & \Leftarrow & (2, n - K + 1) \cdots \Leftarrow (2, n - 2). \end{array}$$

For example, for Lie 3-algebras and for $K = 3$, we have the following relationship among the various definitions of semisimplicity:

$$\begin{array}{c} (3, 0) = \text{Filippov-semisimplicity} \\ \Downarrow \\ (2, 0) = \text{Kuzmin-semisimplicity} \Leftarrow (2, 1)\text{-semisimplicity}. \end{array}$$

We have seen that a metric Lie n -algebra which is semisimple in the Kuzmin sense has a non-degenerate Killing form. However, except for the case of $n = 2$, this does not lead to a simple root space decomposition. We also know that the more restrictive notion of Filippov-semisimplicity (i.e. $(n, 0)$ -semisimplicity) is too strong as it implies that the Lie n -algebra is a direct sum of simple factors like $\mathcal{A}_{q, n-q}$ and Abelian ones. What about the other notions of semisimplicity? Does any of them imply a simple structure of the root space?

To proceed, we make the following observation: Naively one may try to relate the Lie algebra $\mathcal{A}_{\langle \bar{h} \rangle}$ (with $\bar{h}_1, \dots, \bar{h}_{n-2} \in \mathcal{A}$) with the Lie algebra living on the D-branes obtained by compactification. However it is certainly too strong, and also unreasonable to demand this to be true for any choice of the elements \bar{h}_i . It turns out, due to the following lemma, a distinguished choice is to take the \bar{h}_i 's to be in a Cartan subalgebra \mathcal{H} of \mathcal{A} .

Lemma 4.2. *Let \mathcal{A} be a Lie n -algebra and \mathcal{H} a Cartan subalgebra. Then \mathcal{H} is also a Cartan subalgebra of the associated Lie $(n - p)$ -algebra $\mathcal{A}_{(\bar{h}_1, \dots, \bar{h}_p)}$ if $\bar{h}_1, \dots, \bar{h}_p \in \mathcal{H}$.*

Proof. That \mathcal{H} is a nilpotent subalgebra of $\mathcal{A}_{(\bar{h}_1, \dots, \bar{h}_p)}$ follows immediately from the fact that \mathcal{H} is a nilpotent subalgebra (in the sense of Filippov) of \mathcal{A} . Next consider x in the normalizer of $N_{(\bar{h}_1, \dots, \bar{h}_p)}(\mathcal{H})$ of \mathcal{H} in $\mathcal{A}_{(\bar{h}_1, \dots, \bar{h}_p)}$, i.e. $[x, \underbrace{\mathcal{H}, \dots, \mathcal{H}}_{n-p-1}, \bar{h}_1, \dots, \bar{h}_p] \in \mathcal{H}$. Due to the property (3.14) of the n -bracket, it follows immediately that $x \in \mathcal{H}$ and hence $N_{(\bar{h}_1, \dots, \bar{h}_p)}(\mathcal{H}) = \mathcal{H}$. Therefore \mathcal{H} is a Cartan subalgebra of $\mathcal{A}_{(\bar{h}_1, \dots, \bar{h}_p)}$. \square

Definition 4.2. Let \mathcal{A} be a Lie n -algebra and \mathcal{H} a Cartan subalgebra. We say that \mathcal{A} is *strong-semisimple* if there exists $\bar{h}_1, \dots, \bar{h}_{n-2} \in \mathcal{H}$ such that the Lie algebra $\mathcal{A}_{\langle \bar{h} \rangle}$ is semisimple, i.e. \mathcal{A} is $(2, n - 2)$ -semisimple with respect to a choice of elements $\bar{h}_1, \dots, \bar{h}_{n-2} \in \mathcal{H}$.

Since under suitable conditions, a system of multiple M2-branes can always be reduced to a system of multiple D2-branes, one expects a natural connection between the Lie 3-algebraic symmetry of the system of M2-branes and the semisimple Lie algebra of the system of D2-branes. It seems therefore quite reasonable to require that the Lie 3-algebra to be employed in the BLG theory to be strong-semisimple. We conjecture that strong-semisimple Lie 3-algebras is the appropriate kind of Lie 3-algebras that is relevant for describing the symmetry of multiple M2-branes.

In the following, we will analysis the structure of a strong-semisimple Lie n -algebra in more details. In particular we will show that the notion of strong-semisimplicity leads to great simplification of the root space decomposition (3.26), yielding a simple and yet rich structure for the root-space decomposition of a Lie n -algebra, see theorem 4.3 and explicitly (4.16)-(4.19c).

4.2 Generalized Cartan-Weyl n -algebras

Consider a strong-semisimple Lie n -algebra \mathcal{A} and let \mathcal{H} be a Cartan subalgebra. Let $\bar{h}_1, \dots, \bar{h}_{n-2} \in \mathcal{H}$ be a choice of $n-2$ elements such that the Lie algebra $\mathcal{A}_{\langle \bar{h} \rangle}$ is semisimple. Using the fact that the Killing metric $\bar{\kappa}$ on $\mathcal{A}_{\langle \bar{h} \rangle}$ is non-degenerate⁵, one can establish the following important property for the structure of the root space decomposition for a strong-semisimple Lie n -algebra.

Theorem 4.3. *Let \mathcal{A} be a Lie n -algebra which is strong-semisimple, then the root space decomposition for \mathcal{A} takes the form*

$$\mathcal{A} = \mathcal{H} \oplus \underbrace{\left(\bigoplus_{\alpha \in \Delta_{\mathcal{A}}(\mathcal{H})} \mathcal{A}^\alpha \right)}_{\text{dimension } 1}, \quad (4.7)$$

where $\mathcal{A}^\alpha = \{y \in \mathcal{A} \mid [y, (h)] = \alpha((h)) y, \text{ for all } (h) \in \mathcal{H}^{\wedge(n-1)}\}$ is of dimension 1. Moreover, the only integral multiples of α which are roots are $0, \pm\alpha$.

Proof. The proof is a slight generalization of the proof of lemma 2.10. Let $\bar{h}_1, \dots, \bar{h}_{n-2} \in \mathcal{H}$ be the set of elements such that $\mathcal{A}_{(\bar{h}_1, \dots, \bar{h}_{n-2})}$ is semisimple. Since the Killing metric $\bar{\kappa}$ is non-degenerate, we can use it in (3.19) and lemma 3.8. Therefore corresponds to each nonzero root α of the Lie n -algebra, one can associate an element $h_{(\alpha; \bar{h}_1, \dots, \bar{h}_{n-2})} \in \mathcal{H}$ such that

$$\bar{\kappa}(h_{(\alpha; \bar{h}_1, \dots, \bar{h}_{n-2})}, h) = \alpha(\bar{h}_1, \dots, \bar{h}_{n-2}, h). \quad (4.8)$$

Moreover as in lemma 3.8 we have elements $e_\alpha, e_{-\alpha} \in \mathcal{A}$ such that

$$[e_\alpha, e_{-\alpha}, \bar{h}_1, \dots, \bar{h}_{n-2}] = \bar{\kappa}(e_{-\alpha}, e_\alpha) h_{(\alpha; \bar{h}_1, \dots, \bar{h}_{n-2})} (-1)^n. \quad (4.9)$$

Next, as in the proof of lemma 2.10 we introduce the vector space

$$\mathcal{R} := \mathbb{C}e_\alpha \oplus \mathbb{C}h_{(\alpha; \bar{h}_1, \dots, \bar{h}_{n-2})} \oplus \sum_{k=1}^{\infty} \mathcal{A}^{-k\alpha}. \quad (4.10)$$

It is

$$[e_\alpha, \bar{h}_1, \dots, \bar{h}_{n-2}, h] = \alpha(\bar{h}_1, \dots, \bar{h}_{n-2}, h) e_\alpha, \quad (4.11)$$

$$[h_{(\alpha; \bar{h}_1, \dots, \bar{h}_{n-2})}, \bar{h}_1, \dots, \bar{h}_{n-2}, h] = 0, \quad [\mathcal{A}^{-k\alpha}, \bar{h}_1, \dots, \bar{h}_{n-2}, h] \subset \mathcal{A}^{-k\alpha}, \quad (4.12)$$

⁵It has a simple relation with the Killing form on the Lie n -algebra \mathcal{A}

$$\bar{\kappa}(x, y) = \kappa(x, \bar{h}_1, \dots, \bar{h}_{n-2}; y, \bar{h}_1, \dots, \bar{h}_{n-2}), \quad x, y \in \mathcal{A}. \quad (4.6)$$

for all $h \in \mathcal{H}$. Therefore

$$\text{tr}_{\mathcal{R}}(R_{(\bar{h}_1, \dots, \bar{h}_{n-2}, h)}) = \alpha(\bar{h}_1, \dots, \bar{h}_{n-2}, h) [1 - n_{-\alpha} - 2n_{-2\alpha} - \dots]. \quad (4.13)$$

Finally we notice that for $h = h_{(\alpha; \bar{h}_1, \dots, \bar{h}_{n-2})}$, we can use (4.9) to obtain that $R_{(\bar{h}_1, \dots, \bar{h}_{n-2}, h)} \sim [R_{(e_\alpha, \bar{h}_1, \dots, \bar{h}_{n-2})}, R_{(e_{-\alpha}, \bar{h}_1, \dots, \bar{h}_{n-2})}]$ and hence its trace is zero. Moreover (4.8) implies that for this h , $\alpha(\bar{h}_1, \dots, \bar{h}_{n-2}, h) = \bar{\kappa}(h, h) \neq 0$, therefore we obtain $n_{-\alpha} = 1, n_{-2\alpha} = \dots = 0$ and hence our claim. \square

The theorem 4.3 states that a strong-semisimple Lie n -algebra has a basis of generators consisting of a number of step generators E^α whose roots are non-degenerate and a number of generators H_I which spans the Cartan subalgebra \mathcal{H} . The n -brackets expressed in terms of these generators take the form

$$[H_{I_1}, \dots, H_{I_n}] = L_{I_1 \dots I_n}^M H_M, \quad (4.14)$$

$$[H_{I_1}, \dots, H_{I_{n-1}}, E^\alpha] = \alpha_{I_1 \dots I_{n-1}} E^\alpha, \quad (4.15)$$

together with the other relations that follow from (3.14). Here $L_{I_1 \dots I_n}^M$ is a set of constants such that \mathcal{H} form a Cartan subalgebra. In general for $n \geq 3$, the Cartan subalgebra is non-abelian. The brackets are further constrained if there is an invariant metric. For the first nontrivial case of $n = 3$, the 3-brackets of a metric strong-semisimple Lie 3-algebra take the form:

$$[H_I, H_J, H_K] = L_{IJK}^M H_M, \quad (4.16)$$

$$[H_I, H_J, E^\alpha] = \alpha_{IJ} E^\alpha, \quad (4.17)$$

$$[H_I, E^\alpha, E^\beta] = \begin{cases} \alpha_{IK} g^{KL} H_L, & \text{if } \alpha + \beta = 0, \\ g_I(\alpha, \beta) E^{\alpha+\beta}, & \text{if } \alpha + \beta \neq 0 \text{ is a root,} \\ 0, & \text{if } \alpha + \beta \text{ is not a root,} \end{cases} \quad (4.18a)$$

$$[H_I, E^\alpha, E^\beta] = \begin{cases} g_I(\alpha, \beta) E^{\alpha+\beta}, & \text{if } \alpha + \beta \neq 0 \text{ is a root,} \\ 0, & \text{if } \alpha + \beta \text{ is not a root,} \end{cases} \quad (4.18b)$$

$$[H_I, E^\alpha, E^\beta] = \begin{cases} 0, & \text{if } \alpha + \beta \text{ is not a root,} \end{cases} \quad (4.18c)$$

$$[E^\alpha, E^\beta, E^\gamma] = \begin{cases} -g_K(\alpha, \beta) g^{KL} H_L, & \text{if } \alpha + \beta + \gamma = 0, \\ c(\alpha, \beta, \gamma) E^{\alpha+\beta+\gamma}, & \text{if } \alpha + \beta + \gamma \neq 0 \text{ a root,} \\ 0, & \text{if } \alpha + \beta + \gamma \text{ is not a root.} \end{cases} \quad (4.19a)$$

$$[E^\alpha, E^\beta, E^\gamma] = \begin{cases} c(\alpha, \beta, \gamma) E^{\alpha+\beta+\gamma}, & \text{if } \alpha + \beta + \gamma \neq 0 \text{ a root,} \\ 0, & \text{if } \alpha + \beta + \gamma \text{ is not a root.} \end{cases} \quad (4.19b)$$

$$[E^\alpha, E^\beta, E^\gamma] = \begin{cases} 0, & \text{if } \alpha + \beta + \gamma \text{ is not a root.} \end{cases} \quad (4.19c)$$

Within this framework, we see that Cartan-Weyl n -algebras are just special form of strong-semisimple metric Lie n -algebras in the case that the Cartan subalgebra is Abelian. Therefore to be uniform with terminology, we will call strong-semisimple metric Lie n -algebras as *generalized Cartan-Weyl n -algebras*.

Given its possible relevance for describing multiple M2-branes and also due to its naturalness from a pure mathematical point of view, it is an interesting question to classify the generalized Cartan-Weyl n -algebras, particularly for the case $n = 3$. To do this, one needs to solve for the consistency conditions similar to those performed for

the Cartan-Weyl 3-algebras where $L_{IJK}^M = 0$ (see appendix A of [1]). This is a more difficult task and we will leave it for further investigation. In the next subsection, we will consider a certain special class of generalized Cartan-Weyl 3-algebras where it is possible to obtain a rather explicit classification.

4.3 A special class of generalized Cartan-Weyl 3-algebras

If one examines carefully our construction in the appendix A of [1] for the Cartan-Weyl 3-algebras, one can see that we have indeed only used the following relations in our consistency checks,

$$[[H_I, H_J, H_K], H_J, E^\alpha] = 0, \quad (4.20)$$

$$[[H_I, H_J, H_K], E^\alpha, E^\beta] = 0, \quad \text{for any } H\text{'s and } E\text{'s.} \quad (4.21)$$

No where had we used $[H_I, H_J, H_K] = 0$. This means our results obtained in [1] are also valid for a special kind of generalized Cartan-Weyl 3-algebras whose Cartan subalgebra is non-abelian but satisfies (4.20) and (4.21). We observe that when the metric is non-degenerate, the conditions (4.20) and (4.21) are not independent. In fact we have:

Lemma 4.4. *The condition (4.21) follows from (4.20) if the metric g_{IJ} is non-degenerate.*

Proof. To prove this, take two arbitrary roots α, β with $\alpha + \beta$ being a nonzero root. Since $\alpha \neq 0$, it means there exists some H_1, H_2 such that $\alpha_{12} \neq 0$, therefore one can write $E^\alpha = [H_1, H_2, E^\alpha]/\alpha_{12}$. Denote $\Delta = [H_I, H_J, H_K]$ for arbitrary I, J, K . The fundamental identity $[[\Delta, H_1, E^\beta], H_2, E^\alpha] = [[\Delta, H_2, E^\alpha], H_1, E^\beta] + [\Delta, [H_1, H_2, E^\alpha], E^\beta] + [\Delta, H_1, [E^\beta, H_2, E^\alpha]]$ together with the property (4.20) implies that

$$[\Delta, E^\alpha, E^\beta] = 0, \quad \text{for } \alpha + \beta \neq 0. \quad (4.22)$$

Next using the invariance of the metric, we have

$$\langle [\Delta, E^\alpha, E^{-\alpha}], H_L \rangle = -\langle E^{-\alpha}, [\Delta, E^\alpha, H_L] \rangle = 0, \quad \text{for arbitrary } H_L. \quad (4.23)$$

Moreover, one can show that

$$\langle [E^\alpha, E^{-\alpha}, \Delta], E^\delta \rangle = 0, \quad \text{for arbitrary } \delta. \quad (4.24)$$

This is so because we have $\langle [E^\alpha, E^{-\alpha}, \Delta], E^\delta \rangle = -\langle E^\alpha, [E^\delta, E^{-\alpha}, \Delta] \rangle = 0$ for arbitrary $\delta \neq \alpha$; while for $\delta = \alpha$, we can write $\langle [E^\alpha, E^{-\alpha}, \Delta], E^\delta \rangle = -\langle E^{-\alpha}, [E^\alpha, E^\delta, \Delta] \rangle = 0$. Therefore $\langle [E^\alpha, E^{-\alpha}, \Delta], x \rangle = 0$ for all $x \in \mathcal{A}$. Since the metric is non-degenerate, hence our claim. \square

As a result, as long as (4.20) holds, the explicit results of classification obtained in section 3 of [1] remain valid. To see more clearly the meaning of the condition (4.20), let us denote the H 's which appears on the right hand side of $[H_I, H_J, H_K]$ as $h^{(0)}$, then (4.20) is satisfied if

$$[h^{(0)}, H_J, E^\alpha] = 0, \quad \text{for all } E^\alpha \text{ and } H_J. \quad (4.25)$$

Examining our results obtained in [1], e.g. for the general index m case, one see that none of the $u^{(i)}$, $H_{\hat{I}_{(i)}}$ or $H_{\hat{I}_{(\lambda)}}$ satisfies the relation (4.25). However since $v^{(i)}$'s are central in the algebra, it is obvious that our results in section 3 of [1] won't be altered if v 's appears on the right hand side of $[H_I, H_J, H_K]$. This corresponds to a central extension of the Cartan-Weyl 3-algebra, $[H_I, H_J, H_K] = \sum_i L_{IJK}{}^i v^{(i)}$.

There is also another straightforward way one can generalize our results obtained in [1] so that it is applicable for generalized Cartan-Weyl 3-algebras satisfying the condition (4.20). In our check of the consistency conditions in appendix A there, one could have started with a larger set of Cartan generators

$$\{H_I\} = \{H_{\hat{I}}, H_a, h_{\hat{3}}^{(0)}\}, \quad (4.26)$$

where $h_{\hat{3}}^{(0)} \in E_0$ is an additional set of generators with positive norms and appear on the right hand side of $[H_I, H_J, H_K]$

$$[H_I, H_J, H_K] = \sum_l L_{IJK}{}^l v^{(l)} + \sum_{\hat{3}} K_{IJK}{}^{\hat{3}} h_{\hat{3}}^{(0)} \quad (4.27)$$

and satisfies the relation (4.25). Then one can immediately see that, with the relation $[H_I, H_J, H_K] = 0$ replaced with the relation (4.27), all our results obtained in the appendix A and in section 3 of [1] remain valid. The coefficient $L_{IJK}{}^1$ is arbitrary while the coefficient $K_{IJK}{}^{\hat{3}}$ is constrained so that the Cartan subalgebra is nilpotent. We call this a *special generalized Cartan-Weyl 3-algebra*.

We remark that in [17], the general form of metric Lie 3-algebras with a maximally isotropic center with arbitrary index is obtained. In our notation, their results (equation (6) there) read

$$[u^{(i)}, u^{(j)}, u^{(k)}] = \sum_l L_{ijk}{}^l v^{(l)} + \sum_{\hat{\delta}} K_{ijk}{}^{\hat{\delta}} h_{\hat{\delta}}^{(0)}, \quad (4.28)$$

$$[u^{(i)}, u^{(j)}, h_{\hat{3}}^{(0)}] = - \sum_k K_{ijk}{}^{\hat{3}} v^{(k)}, \quad (4.29)$$

$$[H_{\hat{I}_i}, u^{(j)}, u^{(k)}] = 0 \quad \text{etc.} \quad (4.30)$$

This is in fact a special case of our special generalized Cartan-Weyl 3-algebras where the relation (4.27) is solved with a special form of the $L_{IJK}{}^1$ and $K_{IJK}{}^{\hat{3}}$ giving (4.28)-(4.30). Therefore this class of Lie 3-algebras is contained within our framework of generalized Cartan-Weyl 3-algebras.

Let us now look at the question of having a \mathcal{A}_4 embedding. In [1] we have shown that the Cartan-Weyl 3-algebras of any index $m > 0$ generally do not contain a \mathcal{A}_4 subalgebra

$$[X^{\mathcal{P}}, X^{\mathcal{Q}}, X^{\mathcal{R}}] = i\epsilon^{\mathcal{P}\mathcal{Q}\mathcal{R}\mathcal{S}} X^{\mathcal{S}} \quad (4.31)$$

and hence these Lie 3-algebras cannot contain any fuzzy S^3 solution. What about Lie 3-algebras with maximally isotropic centers? The answer is again negative. To see this, let us write $X^{\mathcal{P}} = X^{\mathcal{P}}_{u^{(i)}} u^{(i)} + X^{\mathcal{P}}_{v^{(i)}} v^{(i)} + X^{\mathcal{P}}_{\mathfrak{z}} h_{\mathfrak{z}}^{(0)} + X^{\mathcal{P}}_{+\mu} x_+^{(\mu)} + X^{\mathcal{P}}_{-\mu} x_-^{(\mu)} + X^{\mathcal{P}}_{\alpha} T^{\alpha}$ and note that $u^{(i)}$ never appears on the RHS of the 3-brackets and so $X^{\mathcal{P}}_{u^{(i)}} = 0$ immediately. It follows that the modifications in (4.28)-(4.30) are not effective. Therefore there is no nonvanishing solution to (4.31) for the class of metric Lie 3-algebras with a maximally isotropic centers. This includes the Lorentzian 3-algebra and is consistent with the fact that these kind of BLG theories describe D2-branes rather than uncompactified M2-branes. In the same way one can also check that the special generalized Cartan-Weyl 3-algebras also do not contain any \mathcal{A}_4 subalgebra.

It should be clear from the above discussion that the reason why the Cartan-Weyl 3-algebras or the special generalized Cartan-Weyl 3-algebras do not admit a \mathcal{A}_4 covariant solution is because these Lie 3-algebras all have a set of generators $u^{(i)}$ that do not appear on the right hand side of the 3-brackets. For the generalized Cartan-Weyl 3-algebras, this is no longer the case. Generally, a generalized Cartan-Weyl 3-algebra is specified by, as for a Cartan-Weyl 3-algebra, a set of 2-form roots α and the structure constants $g_I(\alpha, \beta)$, together with the two new sets of coefficients L_{IJK}^M and $c(\alpha, \beta, \gamma)$. In some sense, the function $c(\alpha, \beta, \gamma)$ is the structure that characterize a genuine Lie 3-brackets. However as we have seen in our analysis in [1], it cannot be turned on unless the coefficient L_{IJK}^H is also turned on.

With the activation of L_{IJK}^H , it should be possible to solve (4.31) nontrivially and hence allow the embedding of \mathcal{A}_4 as subalgebra. If this is really the case, the class of generalized Cartan-Weyl 3-algebras will then be the most natural extension of the class of semisimple Lie algebras, much the same as a semisimple Lie algebra always contain the $su(2)$ as a subalgebra. This is of course crucial to the construction of fuzzy sphere solution. This issue is under investigation.

5. Discussions

In this paper, we have proposed a natural reduction condition which connects the Lie algebra which appears in the description of multiple D2-branes and the Lie 3-algebra which appears in the description of multiple M2-branes. Furthermore, we have shown that this reduction condition implies a great simplification of the root space decomposition of the Lie 3-algebras, leading to the notion of a generalized Cartan-Weyl 3-algebras. A generalized Cartan-Weyl 3-algebra is very similar to a Cartan-Weyl 3-algebra, but

with the important difference that its Cartan subalgebra is non-abelian in general. This is a new feature in the theory of Lie n -algebras which does not show up for the special case of $n = 2$.

When the Cartan subalgebra is Abelian, a complete classification of the Cartan-Weyl 3-algebras has been obtained in [1]. It will be interesting to classify the generalized Cartan-Weyl 3-algebras in general; and to work out the specific conditions on $L_{IJK}{}^M$ so that one have embedding of the Lie 3-algebra \mathcal{A}_4 .

In this paper, we have been mostly concerned with the enhanced Lie 3-algebra symmetry of multiple M2-branes. The description of the enhanced gauge symmetry of multiple M5-branes is even more mysterious. Taking the BLG theory with an infinite dimensional Poisson bracket as the underlying Lie 3-algebra, a new formulation of a single M5 brane has been proposed recently [19]. The studies of the gauge symmetry in this description and its resolution may serve to teach us something about the enhanced symmetry of multiple M5-branes, see for example [20–22] for some discussions. Another approach is to study the dynamics of the underlying multiple self-dual strings of the M5-brane theory using the multiple open M2-branes system [7, 23]. A third approach is through the use of the quantum geometry of M5-brane in the presence of a constant C -field discovered in [24]⁶. As noted there, since the Lie algebraic structure describing the symmetry of multiple D-branes already appeared in the theory of a single D-brane when a B -field is present, the quantum geometry of a single M5-brane in the presence of C -field should tell us something about the gauge symmetry structure of multiple M5-branes without C -field. Very recently, an understanding [26] of the quantum geometry of [24] has been achieved in terms of a quantization of the Nambu brackets [27]. It will be very interesting to understand how gauge symmetry is realized in this kind of quantum geometry. It will be helpful to be able to derive the Nambu dynamics from a fundamental well controlled description [28].

Based on the analysis of the entropy counting for M2-branes, it was suggested in [29] that the Lie 3-algebra which is relevant for the description of multiple M2-branes is given by a quantum Nambu bracket. Combined with the arguments here, it means the quantum Nambu bracket should be strong-semisimple. It will be interesting to see if this is the same as the class of generalized Cartan-Weyl 3-algebras which allows \mathcal{A}_4 embedding. The quantization of Nambu bracket is an intriguing problem of great difficulty. See [30–32] for some further discussions on this issue.

⁶The supersymmetric coupling of multiple M2-branes to a non-constant C -field background has been constructed recently in [25].

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